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The Hop domination number of comb product graphs

*D.Anusha, [1]S.Joseph Robin
*Department of Mathematics,
Arunachala Hitech College of Engineering,
Mullanganavilai,Karungal-629 195, India.
anushasenthil84@gmail.com
[1] Department of Mathematics,
Scott Christian College, Nagercoil-629 003, India.
dr.robinscc@gmail.com

Abstract

A set $S \subseteq V$ of a graph G is a hop dominating set of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u,v) = 2$. The minimum cardinality of a hop dominating set of G is called the *hop domination number* and is denoted by $\gamma_h(G)$. Any hop dominating set of order $\gamma_h(G)$ is called γ_h-set of G. In this paper we studied the concept of the hop domination number of comb product of some standard graphs.

Keywords: hop domination number, domination number, comb product.

AMS Subject Classification: 05C12, 05C69.

1. Introduction

By a graph $G = (V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [4]. For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in G / uv \in E(G)\}$. The degree of a vertex $v \in V$ is $deg(v) = |N(v)|$. If $e = \{u,v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then we call e a pendant edge or end edge, u a leaf or end vertex and v a support. A vertex of degree $n - 1$ is called a universal vertex. The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An $u - v$ path of length $d(u,v)$ is called a u-v geodesic. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. For two vertices u and v, the closed interval $I[u,v]$ consists of u and v together with all vertices lying on some u-v geodesic. For a set $S \subseteq V(G)$, in the interval $I_G[S]$ is the union of all $I_G[u,v]$ for $u,v \in S$.

A set $D \subset V$ is a *dominating set* of G if every vertex $v \in V - D$ is adjacent to some vertex in D. A dominating set D is said to be *minimal* if no subset of D is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. The domination number of a graph was studied in [6]. A set $S \subseteq V$ of a graph G is a hop dominating set (hd-set, in short) of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hd-set of G is called the hop domination number and is denoted by $\gamma_h(G)$. Any hd-set of order $\gamma_h(G)$ is called γ_h-set of G. The hop domination number of a graph was studied in [1-3,7-9]. The dominating concept have interesting applications in social networks. By applying the hop dominating concept, we can improve the privacy in social networks.

Let G and H be two connected graphs. Let o be a vertex of H. The comb product between G and H denoted by $G \rhd H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying the i^{th} -copy of H at the vertex o to the i^{th} -vertex of G. By the definition of comb product, we can say that $V(G \rhd H) = \{(a, u): a \in V(G), u \in V(H)\}$ and $(a, u)(b, v) \in E(G \rhd H)$ whenever $a = b$ and $uv \in E(H)$ or $ab \in E(G)$ and $u = v = o$. That concepts were studied in [5].

2. Hop domination number of comb product graphs

Theorem 2.1. Let H=P_{n_1} be the path of order n_1 and K=C_{n_2} be the cycle of order n_2 . Then

$$\gamma_h(H \rhd K) = \begin{cases} n_1 & if \quad n_2 = 4 \; or \; 5 \\ n_1 \left\lceil \frac{n_2}{3} \right\rceil & if \quad n_2 = 6r \; or \; 6r + s, 1 \le s \le 3 \\ n_1 \left\lfloor \frac{n_2}{3} \right\rfloor & if \quad n_2 = 6r + 4 \; or \; 6r + 5, r \ge 1 \end{cases}.$$

Proof: Let $V(H) = \{v_1, v_2, \dots, v_{n_1}\}$ and $V(K) = \{u_1, u_2, \dots, u_{n_2}\}$. Let $V(K_i)=\{u_{i,1}, u_{i,2}, \dots, u_{i,n_2}\}$ be the ith-copy of K and $u_{i,1} (i \le i \le n_1)$ be the root vertex of $G = H \rhd K$.

Case 1: 4 $\le n_2 \le 5$. Let S be a γ_h-set of G. It is easily observed that each root vertex belongs to S.Then $\gamma_h(G) \ge n_1$. Since $S=\{u_{1,1}, u_{2,1}, u_{3,1}, \dots, u_{n_1,1}\}$ is the only γ_h-set of G so that $\gamma_h(G)=n_1$.

Case 2:$n_2 \ge 6$.

Case 2a: $n_2 = 6r$. Let S=$\{u_{i,1}, u_{i,4}, u_{i,7}, u_{i,10}, \ldots, u_{i,6r-2}\}$. Then S is the hop dominating set of G so that $\gamma_h(G) \leq n_1 \left\lceil \frac{n_2}{3} \right\rceil$. We have to prove that $\gamma_h(G) = n_1 \left\lceil \frac{n_2}{3} \right\rceil$. On the contrary, suppose that $\gamma_h(G) \leq n_1 \left\lceil \frac{n_2}{3} \right\rceil$-1. Then there exists a γ_h-set S' of G such that $|S'| \leq n_1 \left\lceil \frac{n_2}{3} \right\rceil$-1. Hence there exists a $x \in V \backslash S'$ such that $d(x,y) \geq 3$, where $y \in S'$. Therefore S' is not a hop dominating set of G, which is a contradiction. Hence $\gamma_h(G) = n_1 \left\lceil \frac{n_2}{3} \right\rceil$.

Case 2b: $n_2 = 6r + 1$ or $6r + 2$ or $6r + 3$. Let T=$\{u_{i,1}, u_{i,4}, u_{i,10}, \ldots, u_{i,6r-2}\}$. $\{u_{i,5}, u_{i,11}, \ldots, u_{i,6r-1}\}$. Then as in Case 2a, we can prove that T is a γ_h-set of G so that $\gamma_h(G) = n_1 \left\lceil \frac{n_2}{3} \right\rceil$.

Case 2c: $n_2 = 6r + 4$ or $6r + 5$. Let W=$\{u_{i\,1}, u_{i\,6}, u_{i\,12}, \ldots, u_{i\,6r}\} \cup \{u_{i,7}, u_{i,13}, \ldots, u_{i,6r+1}\}$. Then as in Case 2a, we can prove that W is a γ_h-set of G so that $\gamma_h(G) = n_1 \left\lceil \frac{n_2}{3} \right\rceil$. ■

Theorem 2.2. Let H=P_{n_1} be the path of order $n_1 \geq 2$ and K=P_{n_2} be the path of order $n_2 \geq 3$.

$$\text{Then}\gamma_h(H \rhd K) = \begin{cases} n_1 & if \;\; n_2 \geq 3 \\ 2n_1 & if \;\; n_2 = 4 \; or \; 5 \\ n_1 \left\lceil \frac{n_2}{3} \right\rceil & if \;\; n_2 = 6r \; or \; 6r + s, 1 \leq s \leq 3 \\ n_1(\left\lceil \frac{n_2}{3} \right\rceil + 1) & if \;\; n_2 = 6r + 4 \; or \; 6r + 5, r \geq 1 \end{cases}$$

Proof: Let $V(H) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(K) = \{u_1, u_2, \ldots, u_{n_2}\}$. Let $V(K_i)$=$\{u_{i,1}, u_{i,2}, \ldots, u_{i,n_2}\}$ be the i^{th}-copy of K and $u_{i,1}$($i \leq i \leq n_1$) be the root vertex of G= $H \rhd K$.

Case 1: $1 \leq n_2 \leq 3$

Case 1a: $n_2 \geq 3$.Then S=$\{u_{1,1}, u_{2,1}, u_{3,1}, \ldots, u_{n_{1},1}\}$ is the only γ_h-set of G so that $\gamma_h(G)$= n_1.

Case 1b: $4 \leq n_2 \leq 5$

Then S=$\{u_2, u_3\}$ is the only γ_h-set of G so that $\gamma_h(G)$=2n_1.

Case 2: $n_2 \geq 6$.

Case 2a: $n_2 = 6r$. Let S=$\{u_{i,3}, u_{i,9}, \ldots, u_{i,6r-3}\} \cup \{u_{i,4}, u_{i,10}, \ldots, u_{i,6r-2}\}$. Then S is the hop dominating set of G so that $\gamma_h(G) \leq n_1 \left\lceil \frac{n_2}{3} \right\rceil$. We have to prove that $\gamma_h(G) = n_1 \left\lceil \frac{n_2}{3} \right\rceil$. On the contrary, suppose that $\gamma_h(G) \leq n_1 \left\lceil \frac{n_2}{3} \right\rceil$-1. Then there exists a γ_h-set S' of G such that $|S'| \leq$

$n_1\left\lceil\frac{n_2}{3}\right\rceil$-1. Hence there exists a $x \in V\backslash S'$ such that *d(x,y)≥3,* where $y \in S'$. Therefore S' is not a hop dominating set of *G,* which is a contradiction.Hence $\gamma_h(G) = n_1\left\lceil\frac{n_2}{3}\right\rceil$.

Case 2b: $n_2 = 6r + 1$ *or* $6r + 2$ *or* $6r + 3$.Let T $= \{u_{i,1}, u_{i,6}, u_{i,12}, \ldots, u_{i,6r}\} \cup \{u_{i,7}, u_{i,13}, \ldots, u_{i,6r+1}\}$. Then *as* in Case 2a, we can prove that *T* is a γ_h-set of *G* so that $\gamma_h(G) = n_1\left\lceil\frac{n_2}{3}\right\rceil$.

Case 2c: $n_2 = 6r + 4$ .Let W $= \{u_{i,1}, u_{i,4}, u_{i,7}, \ldots, u_{i,6r+4}\}$. Then as in Case 2a,we can prove that *W* is a γ_h-set of *G* so that $\gamma_h(G) = n_1\left(\left\lceil\frac{n_2}{3}\right\rceil + 1\right)$.

Case 2d: $n_2 = 6r + 5$ .Let *Z*=*W* $\cup \{u_{i,6r+5}\}$. Then as in Case 2a,we can prove that *Z* is a γ_h-set of *G* so that $\gamma_h(G) = n_1\left(\left\lceil\frac{n_2}{3}\right\rceil + 1\right)$. ■

Theorem 2.3. Let *H* and *K* be two connected graphs. Then $\gamma_h(H \rhd K) \le |V(H)|.\gamma_h(K)$

Proof. Let $V(H) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(K) = \{u_1, u_2, \ldots, u_{n_2}\}$.

Let *V(K_i)*=$\{u_{i,1}, u_{i,2}, \ldots, u_{i,n_2}\}$ $(i \le i \le n_1)$ be the i^{th}-copy of *K*. Without loss of generality, let us assume that $u_{i,1}(i \le i \le n_1)$ be the root vertex of *G*= $H \rhd K$. Let *S* be a γ_h-set of *G*. Then

$\gamma_h(H \rhd K) \le |V(H)|.|S| = |V(H)|.\gamma_h(K)$. ■

Theorem 2.4. Let *H* be a connected graph of order n_1 and and *K* be a connected graph of order n_2 with *d(K)*=2.Then $\gamma_h(H \rhd K) = n_1$.

Proof. Let $V(H) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(K) = \{u_1, u_2\}$. Let *V(K_i)*=$\{u_{i,1}\}$ be the i[th]-copy of *K* and $u_{i,1}(1 \le i \le n_1)$ be the root vertex of *G*= $H \rhd K$. We prove that $\gamma_h(G) = n_1$.On the contrary suppose that $\gamma_h(G) \le n_1 - 1$.Then there exists a γ_h-set S' of G such that $|S'| \le n_1$-1. Hence there exists a $x \in V\backslash S'$ such that *d(x,y)≥3,* where $y \in S'$. Therefore S' is not a hop dominating set of *G,* which is a contradiction. Hence $\gamma_h(H \rhd K) = n_1$. ■

Theorem 2.5. Let *H* be any connected graph of order n_1 and and *K* be a connected graph of order n_2 with *d(K)*=1.Then $\gamma_h(H \rhd K)= \begin{cases} 2 & if \quad n_1 = 3 \\ \left\lceil\frac{n_1}{2}\right\rceil & if \;\; n_1 = 4r \; or \; 4r+1 or \; 4r+3 \\ \left\lceil\frac{n_1}{2}\right\rceil + 1 & if \quad n_1 = 4r+2, r \ge 1 \end{cases}$

Proof: Let $V(H) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(K) = \{u_1, u_2\}$. Let *V(K_i)*=$\{u_{i,1}\}$ be the i[th]-copy of *K* and $u_{i,1}(1 \le i \le n_1)$ be the root vertex of *G*= $H \rhd K$.

Case 1: $n_1 = 3$.Then S=$\{u_{2,1}, u_{3,1}\}$ is the only γ_h-set of G so that

$\gamma_h(G)$=2.

Case 2: : $n_1 \geq 4$.

Case 2a:$n_1 = 4r$. Let S=$\{u_{2,1}, u_{6,1} \dots, u_{4r-2,1}\} \cup \{u_{3,1}, u_{7,1} \dots, u_{4r-1,1}\}$. Then S is the hop dominating set of G so that $\gamma_h(G) \leq \left\lceil \frac{n_1}{2} \right\rceil$. We have to prove that $\gamma_h(G) = \left\lceil \frac{n_1}{2} \right\rceil$. On the contrary, suppose that $\gamma_h(G) \leq \left\lceil \frac{n_1}{2} \right\rceil$-1. Then there exists a γ_h-set S' of G such that $|S'| \leq \left\lceil \frac{n_1}{2} \right\rceil$-1. Hence there exists a $x \in V \backslash S'$ such that $d(x,y) \geq 3$, where $y \in S'$. Therefore S' is not a hop dominating set of G, which is a contradiction. Hence $\gamma_h(G) = \left\lceil \frac{n_1}{2} \right\rceil$.

Case 2b: $n_1 = 4r + 1$ Let $T = S \cup \{u_{4r,1}\}$. Then as in Case 2a,we can prove that T is a γ_h-set of G so that $\gamma_h(G) = \left\lceil \frac{n_1}{2} \right\rceil$.

Case 2c: $n_1 = 4r + 3$ .Let W=$\{u_{2,1}, u_{6,1} \dots, u_{4r+2,1}\} \cup \{u_{3,1}, u_{7,1} \dots, u_{4r+3,1}\}$.
Then as in Case 2a,we can prove that W is a γ_h-set of G so that
$\gamma_h(G) = \left\lceil \frac{n_1}{2} \right\rceil$.

Case 2d: $n_1 = 4r + 2$ .Let $W = \{u_{21}, u_{61} \dots, u_{4r-2\,1}\} \cup \{u_{31}, u_{71} \dots, u_{4r-1\,1}\} \cup \{u_{4r\,1}, u_{4r+1\,1}\}$.
Then as in Case 2a,we can prove that W is a γ_h-set of G so that $\gamma_h(G) = \left\lceil \frac{n_1}{2} \right\rceil + 1$. ■

Corollary 2.6. Let $H = P_{n_1}$ be the path of order n_1 and $K = K_{1,n_2}$ be the path of order n_2 .

Then $\gamma_h(H \rhd K) = \begin{cases} 2 & if \quad n_1 = 3 \\ \left\lceil \frac{n_1}{2} \right\rceil & if \quad n_1 = 4r \; or \; 4r + 1 or \; 4r + 3 \\ \left\lceil \frac{n_1}{2} \right\rceil + 1 & if \quad n_1 = 4r + 2, r \geq 1 \end{cases}$ ■

Theorem 2.7. Let H and K be two connected graphs of orders n_1 and n_2 respectively.Then $\gamma_h(H \rhd K) = 2$ if and only if H is K_2 and $d(K) \leq 2$.

Proof. Let G= $H \rhd K$ and $\gamma_h(G) = 2$. Hence it follows from Theorem2.2, that $n_1=2$.
Therefore $H = K_2$. Since $\gamma_h(G) = 2$, $S = V(H)$. Let $V(K) = \{u_1, u_2, \dots, u_{n_2}\}$ and $V(Ki)$= $\{u_{i,1}, u_{i,2}, \dots, u_{i,n_2}\}(i \leq i \leq n_1)$ be the i^{th} -copy of K. We have to prove that $d(K) \leq 2$. On the contrary, suppost that $d(K) \geq 3$. Let $P : x_1, x_2, \dots, x_k$ $(k \geq 3)$ be a diametral path in K and $P_i : x_{i,1}, x_{i,2}, \dots, x_{i,k}$ $(1 \leq i \leq n_2)$ be a diametral path in K_i. Then there exists $x_{ij}' \in V(P_i)$ such that either $d(x, x_{ij}') \geq 3$ or $d(y, x_{ij}') \geq 3$, which is a contradiction.Therefore $d(K) \leq 2$. ■

Corollary 2.8. Let H and K be two connected graphs of orders n_1 and n_2 respectively.Then $\gamma_h(H \rhd K) = 2$ if and only if H is K_2 and K is K_{1,n_2}. ■

Corollary 2.9. Let H and K be two connected graphs of orders n_1 and n_2 respectively.
Then $\gamma_h(H \rhd K) = 2$ if and only if H is K_2 and K is either C_3 or C_4 or C_5. ■

Theorem 2.10. Let H be a connected graph of order n_1 and K be a connected graph of order

n_2 with d ≥ 3. Then $S \subseteq V(K_i)$, for all i ($i \leq i \leq n_2$).

Proof. We prove that $S \subseteq V(K_i)$, for all i ($i \leq i \leq n_2$). On the contray suppose that $S \not\subset V(K_i)$, for all i ($i \leq i \leq n_2$). Let $P : x_1, x_2, \ldots, x_k$ ($k \geq 3$) be a diametral path in K and $P_i : x_{i,1}, x_{i,2}, \ldots, x_{i,k}$ ($1 \leq i \leq n_2$) be a diametral path in K_i. If $S \subseteq V(H)$, then there exists a $x_{ij} \in K_i$ such that $d(S, x_{ij}) \geq 3$, which is a contradiction. Hence $S \subseteq V(K_i)$, for all i ($i \leq i \leq n_2$). ■

References

[1] D.Anusha, J.John and S.Joseph Robin, Graphs with small and large hop domination numbers, Bulletin of IMVI, 11(3), (2021), 483-489.

[2] D.Anusha and S.Joseph Robin,The geodetic hop domination number of a graph, (communicated).

[3] D.Anusha, J.John and S.Joseph Robin,The geodetic hop domination number of complementary prisms, Discrete Mathematics, Algorithms and Applications, 2021, 2150077, 15 pages.

[4] F.Buckley and F.Harary, Distance in Graph, Addition-Wesly-wood city, CA (1990).

[5] Darmaji and Ridho Alfarisi, On the partition of comb product of path and complete graph, International Computation AIP conf. Proc,1867,020038-1-020038-7: doi:10.1063/14994441

[6] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, Fundamentals of domination in graphs, Marcel Dekker, New York, 1998.

[7] S.M. Meera Rani and T.Hemalatha, Hop graph of a graph, International Journal of Informative and Futuristic Research, 3(9), (2016), 3375-3384.

[8] Michael A.Henning and Nader Jafari Rad, On 2-step and hop dominating sets in graphs, Graphs and Combinatorics, 33,(2),(2017), 1-15.

[9] C. Natarajan and S.K.Ayyaswamy, Hop domination sets in graphs-II, Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica,(ASUOC), 23(2), (2015),187-199.